Najtoplije zahvaljujem **prof. Luki Čelikoviću** na dozvoli da skeniram sažetak predavanja
"Opseg i površina figure omeđene kružnim lukovima"
i objavim na svojim web stranicama.

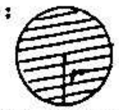
Antonija Horvatek
http://public.carnet.hr/~ahorvate

Luka Čeliković:

OPSEG I POVRŠINA FIGURE OMEĐENE KRUŽNIM LUKOVIMA

Pri određivanju opsega i površina figura omeđenih kružnim lukovima koristit ćemo formule:

KRUG:



$o_k = 2r\pi$ opseg kruga

$p_k = r^2\pi$ površina kruga

KRUŽNI ISJEČAK:

$\ell = \frac{r\pi\alpha^\circ}{180^\circ}$ duljina kružnog luka

$o_{ki} = 2r + \ell$ opseg kružnog isječka

$p_{ki} = \frac{\ell r}{2} = \frac{r^2\pi\alpha^\circ}{360^\circ}$ površina kružnog isječka

KRUŽNI ODSJEČAK:

$o_{ko} = d + \ell$ opseg kružnog odsječka

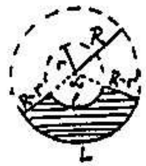
$p_{ko} = p_{ki} - p_{\triangle}$ površina kružnog odsječka

KRUŽNI VIJENAC:

$o_{kv} = 2\pi(R + r)$ opseg kružnog vijenca

$p_{kv} = \pi(R - r)(R + r)$ površina kružnog vijenca

ISJEČAK KRUŽNOG VIJENCA:



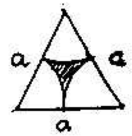
$$\left.\begin{aligned} L &= \frac{R\pi\alpha^\circ}{180^\circ} \\ \ell &= \frac{r\pi\alpha^\circ}{180^\circ} \end{aligned}\right\}$$ duljine kružnih lukova

$o_{ikv} = 2(R - r) + L + \ell$ opseg i.k.v.

$p_{ikv} = \frac{1}{2}(LR - \ell r) = \frac{\pi\alpha^\circ}{360^\circ}(R - r)(R + r)$ povrs. i.k.v.

Odredimo sada opsege i površine slijedećih figura:

Primjer 1.

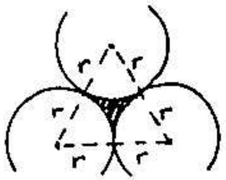


Rješenje:

$$o = \frac{1}{2}\cdot o_{k(r=a/2)} = \frac{a\pi}{2}, \quad p = p_{\triangle(A=B=C=a)} - \frac{1}{2}\cdot p_{k(r=a/2)} = \frac{a^2\sqrt{3}}{4} - \frac{1}{2}\cdot\left(\frac{a}{2}\right)^2\pi =$$

$$= \frac{a^2}{8}(2\sqrt{3} - \pi).$$

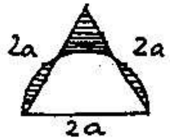
Primjer 2.



Rješenje:

Prema primjeru 1. izlazi $o = r\pi$, $p = (\sqrt{3} - \frac{\pi}{2})r^2$.

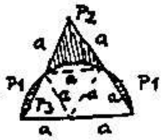
Primjer 3.



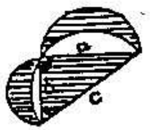
Rješenje:

$o = \frac{1}{2} \cdot o_{k(r=a)} + \frac{2}{3} \cdot o_{\Delta(A=B=C=2a)} = a\pi + 4a = (\pi + 4)a$,

$p = (p_1 + p_2) + p_1 = p_3 + p_1 = p_{ki(r=a, \alpha=60^\circ)} = \frac{1}{6} \cdot p_{k(r=a)} = \frac{1}{6}a^2\pi$



Primjer 4.

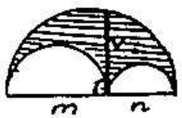


Rješenje:

$o = \frac{1}{2} \cdot o_{k(r=a/2)} + \frac{1}{2} \cdot o_{k(r=b/2)} + \frac{1}{2} \cdot o_{k(r=c/2)} + o_{\Delta(A=a, B=b, C=c)} =$
$= \frac{1}{2}a\pi + \frac{1}{2}b\pi + \frac{1}{2}c\pi + a + b + c = (\frac{\pi}{2} + 1)(a + b + c)$,

$p = \frac{1}{2} \cdot p_{k(r=a/2)} + \frac{1}{2} \cdot p_{k(r=b/2)} - \frac{1}{2} \cdot p_{k(r=c/2)} +$
$+ 2 \cdot p_{\Delta(A=a, B=b, C=c)} = \frac{1}{2} \cdot (\frac{a}{2})^2\pi + \frac{1}{2} \cdot (\frac{b}{2})^2\pi - \frac{1}{2} \cdot (\frac{c}{2})^2\pi + 2 \cdot \frac{ab}{2} =$
$= \frac{\pi}{8}(a^2 + b^2 - c^2) + ab = \frac{\pi}{8}(c^2 - c^2) + ab = ab = 2 \cdot p_{\Delta(A=a, B=b, C=c)}$.

Primjer 5.

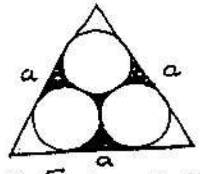


Rješenje:

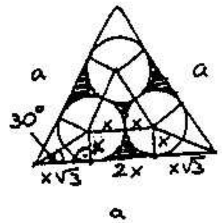
$o = \frac{1}{2} \cdot o_{k(r=(m+n)/2)} + \frac{1}{2} \cdot o_{k(r=m/2)} + \frac{1}{2} \cdot o_{k(r=n/2)} = \frac{1}{2}((m+n)\pi + m\pi + n\pi) =$
$= (m+n)\pi$,

$p = \frac{1}{2} \cdot p_{k(r=(m+n)/2)} - \frac{1}{2} \cdot p_{k(r=m)} - \frac{1}{2} \cdot p_{k(r=n)} =$
$= \frac{1}{2}((\frac{m+n}{2})^2\pi - (\frac{m}{2})^2\pi - (\frac{n}{2})^2\pi) = \frac{mn\pi}{4} = (\frac{v}{2})^2\pi$.

Primjer 6.



Rješenje:

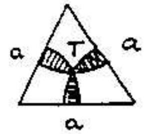


$a = 2x + 2x\sqrt{3} \Rightarrow a = 2x(1 + \sqrt{3}) \Rightarrow x = \frac{a}{2(1+\sqrt{3})} =$
$\Rightarrow x = \frac{\sqrt{3}-1}{4}a$,

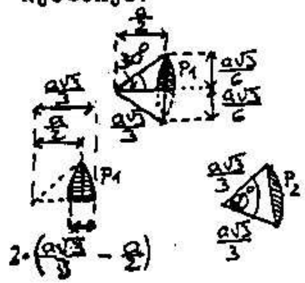
$o = 3 \cdot \frac{2}{3} o_{k(r=x)} + 3 \cdot 2x = 4x\pi + 6x = 2x(2\pi + 3) =$
$= \frac{(\sqrt{3}-1)(2\pi+3)}{2}a$,

$p = 3p_{\square(A=2x, B=x)} + P_{\Delta(A=B=C=2x)} - 3 \cdot \frac{2}{3}P_{k(r=x)} =$
$= 6x^2 + \sqrt{3}x^2 - 2\pi x^2 = x^2(6 + \sqrt{3} - 2\pi) =$
$= \frac{(2-\sqrt{3})(6+\sqrt{3}-2\pi)}{2}a^2 = \frac{9 - 4\sqrt{3} + 2\pi(\sqrt{3}-2)}{2}a^2$.

Primjer 7.



Rješenje:

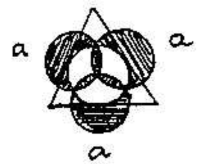


$o = 3 \cdot \frac{1}{6} o_{k(r=a\sqrt{3}/3)} + 3 \cdot 2(\frac{a\sqrt{3}}{3} - \frac{a}{2}) =$
$= \frac{a\pi\sqrt{3}}{3} + a(2\sqrt{3} - 3) = a(\frac{\pi\sqrt{3}}{3} + 2\sqrt{3} - 3)$,

$p_1 = p_2 = p_{ko(r=a\sqrt{3}/3, \alpha=60^\circ)} = p_{ki(r=a\sqrt{3}/3, \alpha=60^\circ)} -$
$- P_{\Delta(A=B=C=a\sqrt{3}/3)} = \frac{1}{6}P_{k(r=a\sqrt{3}/3)} -$
$- P_{\Delta(A=B=C=a\sqrt{3}/3)} = \frac{\pi}{18}a^2 - \frac{\sqrt{3}}{12}a^2 = \frac{2\pi - 3\sqrt{3}}{36}a^2$,

$p = 3p_1 = \frac{2\pi - 3\sqrt{3}}{12}a^2$.

Primjer 8.

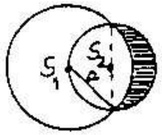


Rješenje:

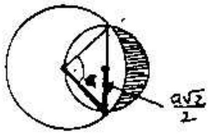
$o = 4 \cdot o_{k(r=a\sqrt{3}/6)} = \frac{4\pi\sqrt{3}}{3}a$,

$p = 3p_{k(r=a\sqrt{3}/6)} - p_{k(r=a\sqrt{3}/6)} = 2p_{k(r=a\sqrt{3}/6)} = \frac{\pi}{6}a^2$.

Primjer 9.



Rješenje:

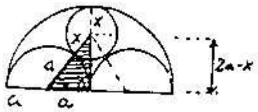


$$o = \frac{1}{4}\cdot o_{k(r=a)} + \frac{1}{2}\cdot o_{k(r=a\sqrt{2}/2)} = \frac{\pi}{2}a + \frac{\pi\sqrt{2}}{2}a = \frac{\pi(1+\sqrt{2})}{2}a,$$

$$p = \frac{1}{2}p_{k(r=a\sqrt{2}/2)} - p_{ko(r=a,\alpha=90^\circ)} = \frac{\pi}{4}a^2 - \left(\frac{\pi}{4}a^2 - \frac{1}{2}a^2\right) = \frac{1}{2}a^2.$$

Primjer 10.

Rješenje:

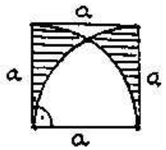


$$(a+x)^2 = a^2 + (2a-x)^2 \Rightarrow x = \frac{2}{3}a,$$

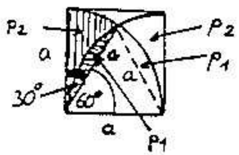
$$o = \frac{1}{2}\cdot o_{k(r=2a)} + o_{k(r=a)} + o_{k(r=2a/3)} = 2a\pi + 2a\pi + \frac{4}{3}\pi a = \frac{16\pi}{3}a,$$

$$p = \frac{1}{2}p_{k(r=2a)} - p_{k(r=a)} - p_{k(r=2a/3)} = \frac{5\pi}{9}a^2.$$

Primjer 11.



Rješenje:



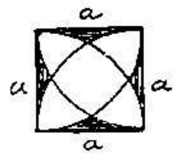
$$o = \frac{1}{2}\cdot o_{k(r=a)} + 2a = a\pi + 2a = (\pi+2)a,$$

$$p_1 = p_{ko(r=a,\alpha=60^\circ)} = p_{ki(r=a,\alpha=60^\circ)} - p_{\Delta(A=B=C=a)} = \frac{\pi}{6}a^2 - \frac{\sqrt{3}}{4}a^2 = \frac{1}{12}(2\pi - 3\sqrt{3})a^2,$$

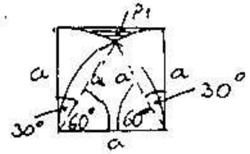
$$p_2 = p_{ki(r=a,\alpha=30^\circ)} - p_1 = \frac{1}{12}(3\sqrt{3} - \pi)a^2,$$

$$p = 2p_2 = \frac{1}{6}(3\sqrt{3} - \pi)a^2.$$

Primjer 12.



Rješenje:

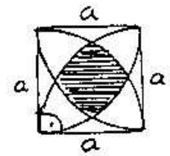


$$o = 4a + \frac{2}{3}\cdot o_{k(r=a)} = 4a + \frac{4\pi}{3}a = \frac{4}{3}(\pi+3)a,$$

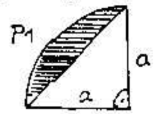
$$p_1 = p_{\square(A=a)} - 2p_{ki(r=a,\alpha=30^\circ)} - p_{\angle(A=B=C=a)} = a^2 - \frac{\pi}{6}a^2 - \frac{\sqrt{3}}{4}a^2 = \frac{1}{12}(12 - 2\pi - 3\sqrt{3})a^2,$$

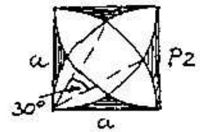
$$p = 4p_1 = \frac{1}{3}(12 - 2\pi - 3\sqrt{3})a^2.$$

Primjer 13.



Rješenje:





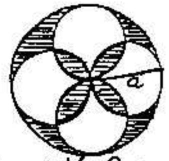
$$o = \frac{1}{3}\cdot o_{k(r=a)} = \frac{2\pi}{3}a,$$

$$p_1 = \frac{1}{4}p_{k(r=a)} - \frac{1}{2}p_{\Delta(A=B=a,\,C=a\sqrt{2})} = \frac{\pi}{4}a^2 - \frac{1}{2}a^2 = \frac{1}{4}(\pi-2)a^2.$$

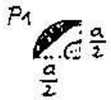
Prema primjeru 12. je $p_2 = \frac{1}{3}(12 - 2\pi - 3\sqrt{3})a^2$, pa iz $4p_1 + p_2 = p_{\square(A=a)} + p$ slijedi

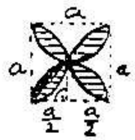
$$p = 4p_1 + p_2 - p_{\square(A=a)} = \frac{1}{3}(3 + \pi - 3\sqrt{3})a^2.$$

Primjer 14.



Rješenje:





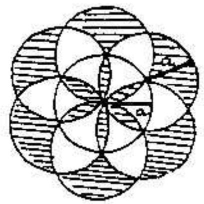
$$o = o_{k(r=a)} + 4\cdot o_{k(r=a/2)} = 6a\pi,$$

$$p_1 = p_{ki(r=a/2,\alpha=90^\circ)} - p_{\Delta(A=B=a/2,\,C=a\sqrt{2}/2)} = \frac{1}{4}p_{k(r=a/2)} - p_{\Delta(A=B=a/2,\,C=a\sqrt{2}/2)} = \frac{1}{4}\left(\frac{a}{2}\right)^2\pi - \frac{1}{2}\left(\frac{a}{2}\right)^2 = \frac{1}{16}(\pi-2)a^2,$$

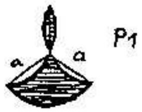
$$p_2 = 8p_1 = \frac{1}{2}(\pi-2)a^2,$$

$$p = p_{k(r=a)} - (4p_{k(r=a/2)} - p_2) + p_2 = a^2\pi - a^2\pi + 2p_2 = 2p_2 = (\pi-2)a^2.$$

Primjer 15.



Rješenje:



$o = 6 \cdot o_{k(r=a)} = 12\pi a.$

$p_1 = p_2 = p_{ki(r=a, \alpha=120^\circ)} = \frac{\pi}{3} a^2,$

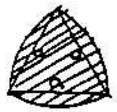
$p = 6p_1 = 2\pi a^2.$

Zadaci za vježbu

Zadatak 1.

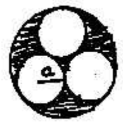
(Rez.: $o = \frac{3(\sqrt{3}-1)\pi}{2} a$, $p = \frac{3(2-\sqrt{3})\pi}{8} a^2$).

Zadatak 2.



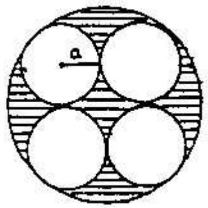
(Rez.: $o = \pi a$, $p = \frac{\pi+\sqrt{3}}{2} a^2$).

Zadatak 3.



(Rez.: $o = \frac{2\pi}{3}(6+\sqrt{3})a$, $p = \frac{\pi}{3}(2\sqrt{3}-1)a^2$).

Zadatak 4.



(Rez.: $o = 2\pi(5+\sqrt{2})a$, $p = \pi(2\sqrt{2}-1)a^2$).

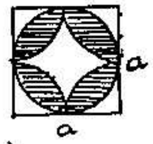
Zadatak 5.

(Rez.: $o = 3\pi a$, $p = \frac{3\pi}{4} a^2$).

Zadatak 6.

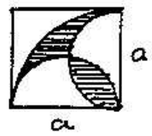
(Rez.: $o = 2\pi(1+\sqrt{2})a$, $p = (\pi(5-2\sqrt{2})-4)a^2$).

Zadatak 7.



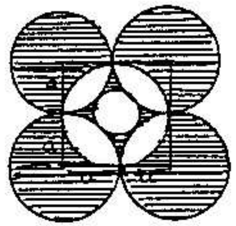
(Rez.: $o = 2\pi a$, $p = \frac{1}{2}(\pi-2)a^2$).

Zadatak 8.



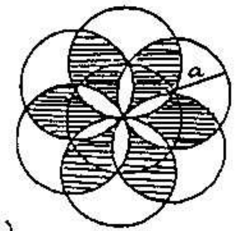
(Rez.: $o = \frac{3\pi}{2} a$, $p = \frac{1}{4}(\pi-2)a^2$).

Zadatak 9.



(Rez.: $o = 2(5\pi+\sqrt{2}-1)a$, $p = (4+\pi(2\sqrt{2}-1))a^2$).

Zadatak 10.



(Rez.: $o = 8\pi a$, $p = 3\sqrt{3}\, a^2$).