

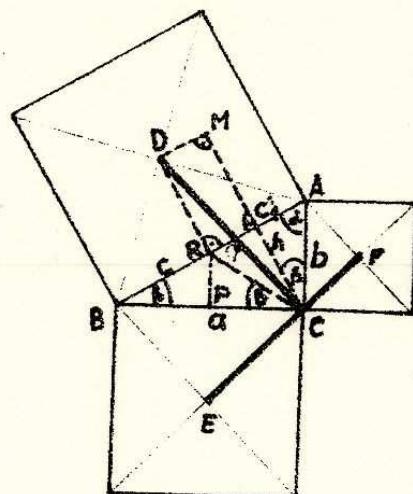
ŠESTI ZADATAK SA PRVIJEG KALIBRA, REPUBLIČKI SUSRET 1989.g.

-1.

$$\begin{aligned} P(x) &= x^4 - 2x^3 + ax^2 + 2x + b = \\ &= (x^2 - 2x - 3)(x^2 + cx + d) = (x+1)(x-3)(x^2 + cx + d) \\ \text{ali je: } P(-1) &= 0 \Rightarrow a+b+1=0 \\ P(3) &= 0 \Rightarrow 9a+b+33=0 \end{aligned}$$

$\Rightarrow a = -4, b = 3$.

-2.



Vidi sliku. Neka je $\alpha \neq \beta$. Tačke E, C i F leže na jednom pravcu, R je polovište dužine \overline{AB} .

$$d(D, R) = \frac{c}{2}, \quad d(C, R) = \sqrt{d(P, R)^2 + d(P, C)^2} = \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2} = \frac{\sqrt{a^2 + b^2}}{2}.$$

Slijedi da je $\triangle CDR$ jednakostrukšen i $\angle CDR = \angle RCD$.

$\angle CDR = \angle DCC_1$ (kutovi s paralelnim kracima).

$$2\angle RCD = 2\angle RCD + 2\angle DCC_1 = 90^\circ - 2\beta = \alpha + \beta - 2\beta = \alpha - \beta \Rightarrow$$

$\angle RCD = \frac{\alpha - \beta}{2}$. Sada je

$$\angle ECD = \angle ECB + \angle BCR + \angle RCD = 45^\circ + \beta + \frac{\alpha - \beta}{2} = 90^\circ.$$

Slijedi da su stranicice EF i CD okomite.

$$d(E, F) = \frac{\sqrt{2}}{2}(a + b).$$

$$P = \frac{ab}{2} = \frac{ch}{2} \Rightarrow h = \frac{ab}{c}.$$

$$d(C_1, R)^2 = d(R, C)^2 - h^2 = \left(\frac{c}{2}\right)^2 - h^2 = \frac{c^2}{4} - \frac{a^2 b^2}{c^2}.$$

$\triangle CMD$ je pravokutan i ima duljine kateta

$$d(C, M) = h + \frac{c}{2} \quad i \quad d(C_1, R) \text{ pa je}$$

$$\begin{aligned} d(C, D)^2 &= d(M, D)^2 + d(C, M)^2 = d(R, C_1)^2 + d(C, M)^2 = \\ &= \frac{c^2}{4} - \frac{a^2 b^2}{c^2} + \left(\frac{ab}{c} + \frac{c}{2}\right)^2 = \frac{c^2 + 2ab}{2} = \frac{(a+b)^2}{2}. \end{aligned}$$

$$\Rightarrow d(C, D) = \frac{\sqrt{2}}{2}(a + b) = d(E, F).$$

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-3. Za pozitivne realne brojeve a, b je

$$(\sqrt{a} - \sqrt{b})^2 \geq 0 \text{ slijedi } \frac{1}{2}(a + b) \geq \sqrt{ab}.$$

Primjenom ove nejednakosti dobivamo

$$\frac{1}{2}\left(\frac{xy}{z} + \frac{yz}{x}\right) \geq y, \quad \frac{1}{2}\left(\frac{yz}{x} + \frac{zx}{y}\right) \geq z, \quad \frac{1}{2}\left(\frac{zx}{y} + \frac{xy}{z}\right) \geq x$$

$$\text{odakle zbrajanjem slijedi } \frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} \geq x + y + z.$$

-4.

Zamijenivši x i y dobivamo:

$$f(y) = f\left(\frac{x+y}{2}\right)g\left(\frac{y-x}{2}\right) + f\left(\frac{y-x}{2}\right)g\left(\frac{x+y}{2}\right).$$

uzući da je f neparna, a g parna funkcija imamo:

$$f\left(\frac{y-x}{2}\right) = -f\left(\frac{x-y}{2}\right), \quad g\left(\frac{y-x}{2}\right) = g\left(\frac{x-y}{2}\right), \quad \text{odnosno}$$

$$f(y) = f\left(\frac{x+y}{2}\right)g\left(\frac{x-y}{2}\right) - f\left(\frac{x-y}{2}\right)g\left(\frac{x+y}{2}\right).$$

z ove i polazne jednakosti dobiva se:

$$f^2(x) - f^2(y) = 4f\left(\frac{x+y}{2}\right)g\left(\frac{x+y}{2}\right)f\left(\frac{x-y}{2}\right)g\left(\frac{x-y}{2}\right). \quad (1)$$

a $y = 0$ u polaznoj jednakosti dobivamo

$$f(x) = 2f\left(\frac{x}{2}\right)g\left(\frac{x}{2}\right)$$

odavde

$$f(x \pm y) = 2f\left(\frac{x \pm y}{2}\right)g\left(\frac{x \pm y}{2}\right). \quad (2)$$

z (1) i (2) slijedi

$$f^2(x) - f^2(y) = f(x+y)f(x-y).$$

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